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ON THE DIFFERENTIABILITY OF THE SOLUTION OF A DIFFERENTIAL EQUATION WITH RESPECT TO A PARAMETER.

BY J. F. RITT.

There will be given in this note a proof of the differentiability of the solution of a single differential equation, of the first order, with respect to a parameter, which does not employ the mechanism of any existence proof. All other treatments of this problem which the writer has seen consist in studying the behavior of the derivatives with respect to the parameter, of the successive approximations to the solution which the existence proof produces. The method usually followed in treatises on analysis is based on the Cauchy-Lipschitz process. Bliss* has recently given a very general treatment of this problem, using the Picard existence proof. Moulton† has also given a proof which is based on the Picard process.

The method given here can also be used in the proof of the differentiability of the solution of a single differential equation with respect to the constant of integration. It does not seem possible to give a simple generalization of the method for systems of differential equations.

Let the equation be

$$\frac{dy}{dx} = f(x, y, a), \tag{1}$$

where $f(x, y, a)$, $f_y(x, y, a)$ and $f_a(x, y, a)$ exist and possess conjoint continuity with respect to all three variables for

$$x_0 \leqq x \leqq x_0 + h, \qquad y_0 - k \leqq y \leqq y_0 + k, \qquad \alpha_1 \leqq a \leqq \alpha_2.$$

We assume also that for every a in the interval (α_1, α_2) there exists a function $y(x)$, defined on the interval $(x_0, x_0 + h)$, satisfying equation (1) in this interval, assuming only values which lie between $y_0 - k$ and $y_0 + k$, and taking on the value y_0 for $x = x_0$. The continuity of $f_y(x, y, a)$ implies the satisfaction of the Lipschitz condition. Hence there will exist only a single solution of (1) for a given value of a, which reduces to y_0 for $x = x_0$.

We shall show that this solution possesses a derivative y_a with respect to a, which satisfies the equation

$$\frac{d}{dx} y_a = f_y y_a + f_a. \tag{2}$$

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* Bulletin of the American Mathematical Society, Oct., 1918.

† Major Moulton's paper was issued in photostat form by the Technical Staff of the Ordnance Department.

Let y_1 be that solution of the equation for the value a_1 of the parameter, which reduces to y_0 for $x = x_0$. We have

$$\frac{d}{dx}(y_1 - y) = f(x, y_1, a_1) - f(x, y, a). \tag{3}$$

Hence if $a_1 \neq a$, and $y_1 \neq y$, we have

$$\frac{d}{dx}\left(\frac{y_1 - y}{a_1 - a}\right) = \frac{f(x, y_1, a_1) - f(x, y, a_1)}{y_1 - y}\,\frac{y_1 - y}{a_1 - a} + \frac{f(x, y, a_1) - f(x, y, a)}{a_1 - a}. \tag{4}$$

Let

$$U = \frac{f(x, y_1, a_1) - f(x, y, a_1)}{y_1 - y}, \qquad V = \frac{f(x, y, a_1) - f(x, y, a)}{a_1 - a}$$

respectively, for $y_1 \neq y$ and for $a_1 \neq a$, and

$$U = f_y(x, y, a_1), \qquad V = f_a(x, y, a),$$

respectively, for $y_1 = y$ and for $a_1 = a$.

Then

$$\frac{d}{dx}\left(\frac{y_1 - y}{a_1 - a}\right) = U\frac{y_1 - y}{a_1 - a} + V, \tag{5}$$

even when $y_1 = y$.

We have

$$U = f_y[x, y + \theta_1(y_1 - y), a_1], \qquad V = f_a[x, y, a + \theta_2(a_1 - a)], \tag{6}$$

where θ_1 and θ_2 lie between 0 and 1.

The function U is conjointly continuous in all of its arguments. This is evident from the definition of U, except in the case of $y_1 = y$. For that case, the continuity follows immediately from the first of equations (6). Similarly, V is continuous in all of its arguments.

We have from equation (5), since $y_1 - y = 0$, for $x = x_0$,

$$\frac{y_1 - y}{a_1 - a} = e^{\int_{x_0}^{x} U dx} \int_{x_0}^{x} e^{\int_{x_0}^{x} -U dx}\, V dx. \tag{7}$$

In this last equation, U and V can be regarded of functions of x, a and a_1, the functions y and y_1 being known. Since the functions U and V are bounded, the ratio of $y_1 - y$ to $a_1 - a$ is also a bounded function of x, a and a_1. Hence, as a_1 approaches a, y_1 approaches y uniformly with respect to x. We see from equations (6) that U and V approach, uniformly with respect to x, the functions $f_y(x, y, a)$ and $f_a(x, y, a)$ respectively. When a continuous function approaches a limit uniformly, the indefinite integral of the function approaches uniformly the indefinite integral of the limit.

Thus the first member of equation (7) approaches a limit y_a, and we have

$$y_a = e^{\int_{x_0}^{x} f_y dx} \int_{x_0}^{x} e^{-\int_{x_0}^{x} f_y dx} f_a dx.$$

Equation (2), which was to be obtained, is found by differentiating this last result.

The method employed above can be used to prove that the solution y is differentiable with respect to its initial value y_0. We shall not go into the details.

WASHINGTON, D. C.,
Apr 1, 1919.